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TRANSLATION

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CAVITIES FILLED WITH A VISCOUS FLUID

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OSCILLATIONS OF A PHYSICAL PENDULUM HAVING CAVITIES FILLED WITH A VISCOUS FLUID

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An asymptotic method has been proposed [1] for investigating the nonstationary motions of viscous incompressible fluids with large Reynolds numbers (N_{Re}), which arise during the oscillation of various solids which either contain a fluid or are emersed in one, and also during the oscillation of liquid volumes having a free surface.

In this article the idea of this method is applied to the investigation of small oscillations of a physical pendulum having cavities completely filled with an incompressible viscous fluid.

1. The pressure of a fluid contained in the cavity of an oscillating pendulum (Fig. 1) is described by the Navier-Stokes equations and the continuity equation:

$$\begin{gathered}\frac{\partial \mathbf{V}'}{\partial t'} + (\mathbf{V}' \cdot \nabla') \mathbf{V}' = \nabla' \varphi' - \nu (\nabla' \times \Omega') \\ \operatorname{div} \mathbf{V}' = 0 \\ \left(\Omega' = \nabla' \times \mathbf{V}', \ \varphi' = -\frac{p}{\rho} - U\right)\end{gathered} \tag{1.1}$$

Here U is the potential of the mass forces acting on the fluid.

At the edge of the cavity the conditions of adhesion of fluid particles to the walls of the cavity should be satisfied. This gives the following boundary conditions:

$$u' = -y'\dot{\theta}, \quad v' = x'\dot{\theta}, \quad w' = 0 \quad \left(\dot{\theta} = \frac{d\theta}{dt'}\right) \tag{1.2}$$

Here θ is the deflection angle of the pendulum from the equilibrium position while u', v', w', are the components of the velocity vector $\mathbf{V}'$.

We assume that

$$\mathbf{V}' = \mathbf{V}'' + \mathbf{V}_0'$$

where $\mathbf{V}_0'$ is the velocity vector of the center of mass of the cavity, and we convert to a new system of coordinates (x", y", z") with its origin at the center of mass of the cavity and with axes parallel to the axes of the stationary system of coordinates (x', y', z').

In the new system of coordinates we will have

$$\frac{\partial \mathbf{V}''}{\partial t'} + (\mathbf{V}'' \cdot \nabla'')\,\mathbf{V}'' = \nabla''\varphi'' - \nu\,(\nabla'' \times \Omega''), \qquad \operatorname{div} \mathbf{V}'' = 0 \tag{1.3}$$

Here

$$\Omega'' = \nabla'' \times \mathbf{V}'', \qquad \varphi'' = -p/\rho - U - (\dot{\mathbf{V}}_0' \cdot \mathbf{r}')$$

At the boundary of the cavity:

$$u'' = -y''\dot{\theta}, \quad v'' = x''\dot{\theta}, \quad w'' = 0 \tag{1.4}$$

We will refer all quantities to a characteristic scale. Let us assume:

$$\begin{gathered} t' = Tt, \quad x'' = Rx, \quad y'' = Ry, \quad z'' = Rz, \quad \mathbf{V}'' = \frac{\alpha}{T} R\mathbf{V} \\ \Omega'' = \frac{\alpha}{T}\Omega, \quad \varphi'' = \alpha\frac{R^2}{T^2}\varphi, \quad \theta = \alpha\vartheta, \quad N_{\mathrm{Re}} = \frac{R^2}{\nu T} \end{gathered} \tag{1.5}$$

where T is the characteristic period of one oscillation, R is the characteristic dimension of the cavity, α is the characteristic amplitude, and N_{Re} is the Reynolds number.

Equations (1.3) and Boundary Conditions (1.4) are written in independent variables in the form:

$$\frac{\partial \mathbf{V}}{\partial t} + \alpha\,(\mathbf{V} \cdot \nabla)\,\mathbf{V} = \nabla\varphi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \Omega), \qquad \operatorname{div} \mathbf{V} = 0 \tag{1.6}$$

At the boundary of the cavity:

$$u = -y\dot{\vartheta}, \quad v = x\dot{\vartheta}, \quad w = 0 \tag{1.7}$$

In the system of equations (1.6) the characteristic amplitude α is a factor for nonlinear terms.

We will consider oscillations with a small amplitude and later linearize Eqs. (1.6), disregarding magnitudes of the order α.

The linearized system of equations appears as follows:

$$\frac{\partial \mathbf{V}}{\partial t} = \nabla\varphi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \Omega), \qquad \operatorname{div} \mathbf{V} = 0 \tag{1.8}$$

The object of this article is to investigate those forms of oscillations for which the solution may be represented in the form:

$$\vartheta(t) = ce^{\sigma t}, \qquad \mathbf{V} = ce^{\sigma t}\mathbf{U}(x, y, z) \tag{1.9}$$

Setting $\varphi = ce^{\sigma t}\Phi(x, y, z), \Omega = ce^{\sigma t}\Psi(x, y, z)$, we have for such motions:

$$\sigma\mathbf{U} = \nabla\Phi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \Psi), \qquad \operatorname{div} \mathbf{U} = 0 \tag{1.10}$$

At the boundary of the cavity:

$$U_x = -y\sigma, \quad U_y = x\sigma, \quad U_z = 0 \tag{1.11}$$

Relationship (1.10) indicates that the vector $\mathbf{U}$ is the sum of the potential and solenoidal vectors. Such a presentation makes it possible to separate the equations, having obtained a separate equation for each unknown function.

Actually, having taken the div of both sides of the first equation of System (1.10) we obtain:

$$\Delta\Phi = 0 \tag{1.12}$$

i.e., a function of Φ which is harmonic in a volume filled with a fluid. Having taken the cure of both sides of the same equation we have:

$$\sigma\Psi = \frac{1}{N_{\mathrm{Re}}}\Delta\Psi \tag{1.13}$$

Thus each of the unknown functions Φ, Ψ satisfies a separate equation.

At the boundary of the cavity these functions are linked by the following boundary conditions:

$$\frac{\partial\Phi}{\partial x} - \frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_z}{\partial y} - \frac{\partial\Psi_y}{\partial z}\right) = -y\sigma^2$$
$$\frac{\partial\Phi}{\partial y} - \frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_x}{\partial z} - \frac{\partial\Psi_z}{\partial x}\right) = x\sigma^2$$
$$\frac{\partial\Phi}{\partial z} - \frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_y}{\partial x} - \frac{\partial\Psi_x}{\partial y}\right) = 0$$

Here Ψ_x, Ψ_y, Ψ_z are components of vector Ψ.

A fourth boundary condition determined at the limit of the projection of vector Ψ onto the normal to the surface of the cavity will be introduced later in a specially selected curvilinear system of coordinates.

Let us consider cavities which appear as figures of revolution whose axes are perpendicular to the plane of oscillations of the body.

We will introduce a curvilinear system of coordinates associated with the surface of the cavity. In the case under consideration it is convenient to introduce the following coordinates: n – distance along the inner normal to the surface of the cavity taken from the surface inward, α – angle which determines the position of the meridian plane, and β – length of arc along the meridian (Fig. 2). The variables in the (x,y,z) coordinate system are associated with the variables n, α, β in the curvilinear coordinate system by the following relationships:

$$x = -\left[r_0 + \int_0^\beta \sin\gamma(\beta)\,d\beta - n\cos\gamma(\beta)\right]\sin\alpha = -[r(\beta) - n\cos\gamma(\beta)]\sin\alpha$$
$$y = \left[r_0 + \int_0^\beta \sin\gamma(\beta)\,d\beta - n\cos\gamma(\beta)\right]\cos\alpha = [r(\beta) - n\cos\gamma(\beta)]\cos\alpha$$
$$z = \int_0^\beta \cos\gamma(\beta)\,d\beta + n\sin\gamma(\beta)$$

Here, r_0 = radius of a circle obtained in the cross-section of the cavity by a plane perpendicular to the axis of symmetry and passing through the center of mass of the cavity;

$r(\beta)$ = radius of the circle formed by points on the surface of the cavity having the coordinate β;

$\gamma(\beta)$ = angle between the tangent to the meridian at the point with coordinate β and the positive direction of the z-axis.

We will write Eq. (1.13) and the boundary conditions in the new coordinate system. For simplicity we will use the notation:

$$\Psi_x = \Psi_1, \quad \Psi_y = \Psi_2 \quad \Psi_z = \Psi_3$$

Then we have in the curvilinear coordinate system:

$$\sigma\Psi_i = \frac{1}{N_{\mathrm{Re}}\,[r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]}\left[\frac{\partial}{\partial\alpha}\left(\frac{1 + n\gamma'(\beta)}{r(\beta) - n\cos\gamma(\beta)}\frac{\partial\Psi_i}{\partial\alpha}\right) + \right.$$
$$\left. + \frac{\partial}{\partial\beta}\left(\frac{r(\beta) - n\cos\gamma(\beta)}{1 + n\gamma'(\beta)}\frac{\partial\Psi_i}{\partial\beta}\right) + \frac{\partial}{\partial n}\left([r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]\frac{\partial\Psi_i}{\partial n}\right)\right] \tag{1.14}$$
$$\left(\gamma'(\beta) = \frac{d\gamma(\beta)}{d\beta}\right) \qquad (i = 1, 2, 3)$$

At the edge of the cavity for n = 0:

$$\frac{\partial\Phi}{\partial n} - \frac{1}{N_{\mathrm{Re}}\,r(\beta)}\left[\frac{\partial\Psi_\beta}{\partial\alpha} - \frac{\partial r(\beta)\,\Psi_\alpha}{\partial\beta}\right] = 0 \tag{1.15}$$

$$\frac{1}{r(\beta)}\frac{\partial\Phi}{\partial\alpha} - \frac{1}{N_{\mathrm{Re}}}\left[\frac{\partial\Psi_n}{\partial\beta} - \frac{\partial(1 + n\gamma'(\beta))\,\Psi_\beta}{\partial n}\right] = r(\beta)\,\sigma^2 \tag{1.16}$$

$$\frac{\partial\Phi}{\partial\beta} - \frac{1}{N_{\mathrm{Re}}\,r(\beta)}\left[\frac{\partial(r(\beta) - n\cos\gamma(\beta))\,\Psi_\alpha}{\partial n} - \frac{\partial\Psi_n}{\partial\alpha}\right] = 0 \tag{1.17}$$

Let us derive the fourth boundary condition. The normal component Ψ_n of the vortex vector **Ψ** is defined in terms of the tangential components U_α and U_β of the velocity vector **U** as follows:

$$\Psi_n = \frac{1}{[r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]}\left[\frac{\partial(1 + n\gamma'(\beta))\,U_\beta}{\partial\alpha} - \frac{\partial(r(\beta) - n\cos\gamma(\beta))\,U_\alpha}{\partial\beta}\right]$$

At the edge of the cavity the vector **U** is known; consequently, its components U_α and U_β are known for n = 0. Differentiation with respect to α and β for n = const is possible, and therefore calculation of the magnitude of Ψ_n for n = 0 presents no particular difficulty. After simple calculations we obtain:

$$\Psi_n = -2 \sin \gamma (\beta) \sigma \tag{1.18}$$

This relationship closes the system of boundary conditions for the unknown functions Φ, Ψ_i.

The quantities Ψ_α, Ψ_β, Ψ_n in Relationships (1.15)-(1.18), are components of vector Ψ in the curvilinear coordinate system. They are connected with the componenets Ψ_i by the relationships:

$$\begin{aligned} \Psi_\alpha &= -\Psi_1 \cos \alpha - \Psi_2 \sin \alpha \\ \Psi_\beta &= -\Psi_1 \sin \gamma (\beta) \sin \alpha + \Psi_2 \sin \gamma (\beta) \cos \alpha + \Psi_3 \cos \gamma (\beta) \\ \Psi_n &= \Psi_1 \cos \gamma (\beta) \sin \alpha - \Psi_2 \cos \gamma (\beta) \cos \alpha + \Psi_3 \sin \gamma (\beta) \end{aligned} \tag{1.19}$$

We will assume that the parameters of the pendulum guarantee a sufficiently large Reynolds number. We will assume:

$$\frac{1}{N_{\mathrm{Re}}} = \varepsilon^2 \tag{1.20}$$

where ε is a small dimensionless parameter.

The idea of constructing a solution for large Reynolds numbers which was presented by Moiseyev [1] is analogous to the idea of constructing a boundary layer. It is assumed that the vortices which exist in the oscillating fluid contained within the pendulum cavity are concentrated mainly in a thin layer at the walls of the cavity. This in turn makes it possible to consider the derivatives of the components of the vector Ψ along the normal to the surface of the cavity to be significantly greater than in the tangential directions. Let us "expand" the independent variable $\underline{n}$. We set

$$n = \varepsilon \eta$$

Solution of the problem formulated will be sought in the form of series to powers of the small parameter ε:

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \ldots, \quad \Psi_i = \frac{1}{\varepsilon} \Psi_{0i} + \Psi_{1i} + \ldots \quad (i = 1, 2, 3) \tag{1.21}$$

Substituting Series (1.21) into the equations and boundary

conditions and equating to zero the sum of the coefficients for the zero power of the parameter ε, we obtain the following problem for the determination of the functions Φ_0 and Ψ_{0i}:

$$\Delta \Phi_0 = 0, \quad \sigma\Psi_{0i} = \frac{\partial^2 \Psi_{0i}}{\partial \eta^2} \qquad (i = 1, 2, 3) \tag{1.22}$$

At the boundary of the cavity, for n = 0:

$$\frac{\partial \Phi_0}{\partial n} = 0, \quad \frac{1}{r(\beta)} \frac{\partial \Phi_0}{\partial \alpha} + \frac{\partial \Psi_{0\beta}}{\partial \eta} = r(\beta)\sigma^2, \quad \frac{\partial \Phi_0}{\partial \beta} - \frac{\partial \Psi_{0\alpha}}{\partial \eta} = 0, \quad \Psi_{0n} = 0 \tag{1.23}$$

From the first relationship of Conditions (1.23) we see that the standard derivative of the harmonic function Φ_0 is zero at the boundary of the cavity. Hence it is possible to consider Φ = const without limiting the generality.

The components of the vector Ψ in the curvilinear system of coordinates are associated with the components of the same vector in the Cartesian coordinate system by Relationships (1.19). These relationships are linear with regard to Ψ_{01}, Ψ_{02}, Ψ_{03} and the coefficients of Ψ_{01}, Ψ_{02}, Ψ_{03} are independent of η; consequently the functions $\Psi_{0\alpha}$, $\Psi_{0\beta}$, and Ψ_{0n} satisfy the same equations as Ψ_{0i}. Thus:

$$\sigma\Psi_{0\alpha} = \frac{\partial^2 \Psi_{0\alpha}}{\partial \eta^2}, \quad \sigma\Psi_{0\beta} = \frac{\partial^2 \Psi_{0\beta}}{\partial \eta^2}, \quad \sigma\Psi_{0n} = \frac{\partial^2 \Psi_{0n}}{\partial \eta^2} \tag{1.24}$$

At the boundary of the cavity these functions satisfy the following boundary conditions:

$$\frac{\partial \Psi_{0\beta}}{\partial \eta} = r(\beta)\sigma^2, \quad \frac{\partial \Psi_{0\alpha}}{\partial \eta} = 0, \quad \Psi_{0n} = 0 \tag{1.25}$$

The general solution of the equation satisfied by $\Psi_{0\alpha}$, $\Psi_{0\beta}$ and Ψ_{0n} has the following form:

$$u = c_1 e^{\sqrt{\sigma}\eta} + c_2 e^{-\sqrt{\sigma}\eta}$$

Let Re $\sqrt{\sigma} > 0$. By assumption, far from the walls of the cavity there are no vortices. Just as is done in boundary-layer theory, we will consider the value $\eta = \infty$ to be the respective internal points

of the cavity sufficiently far from the boundary. Then, in agreement with the assumption concerning the absence of vortices, for $\eta = \infty$ we will have $c_1 = 0$.

The arbitrary constant of integration c_2 is determined from Conditions (1.25). Finally, for the functions $\Psi_{0\alpha}$, $\Psi_{0\beta}$ Ψ_{0n} we obtain:

$$\Psi_{0\alpha} = 0, \quad \Psi_{0\beta} = -r(\beta)\,\sigma\sqrt{\sigma}e^{-\sqrt{\sigma}\eta}, \quad \Psi_{0n} = 0 \tag{1.26}$$

We will limit ourselves to determination of only the first terms of Series (1.21). Within the limits of this accuracy we write the components of the absolute velocity vector $\mathbf{V}'$ in a stationary system of coordinates with its origin at the point of suspension of the pendulum:

$$\begin{aligned} u' &= c\frac{R}{T}\left[l\sigma - r(\beta)\,\sigma\cos\alpha\, e^{-\sqrt{\sigma}\eta} + O(\varepsilon)\right]e^{\sigma t}, \qquad w' = 0 \\ v' &= -c\frac{R}{T}\left[r(\beta)\,\sigma\sin\alpha\, e^{-\sqrt{\sigma}\eta} + O(\varepsilon)\right]e^{\sigma t}, \end{aligned} \tag{1.27}$$

where l is the distance of the center of gravity of the cavity from the suspension axis, relative to the characteristic dimension of the cavity. The resulting solution (1.27) will be asymptotic. Relative to it, we have the following result, proof of which will not be cited here. The difference modulus

$$|\mathbf{V}' - \mathbf{V}_0'| \leqslant O(\varepsilon)$$

Here $\mathbf{V}'$ is the exact solution of the linearized problem and $\mathbf{V}'_0$ is the approximate solution of this problem obtained by us.

Thus, if the Reynolds number is sufficiently high, the approximate solution assures good accuracy.

2. For a final solution of the problem it is necessary to determine the as yet unknown constant σ. For this it is necessary to use the equation of pendulum oscillations which may be obtained using the theorem concerning the variation of angular momentum of a system. We have:

$$\frac{d}{dt}\int_{D_1}(\mathbf{r}'\times\rho_1\mathbf{V}_1')\,d\tau+\frac{d}{dt}\int_{D}(\mathbf{r}'\times\rho\mathbf{V}')\,d\tau=\int_{D_1}(\mathbf{r}'\times\rho_1\mathbf{g})\,d\tau+\int_{D}(\mathbf{r}'\times\rho\mathbf{g})\,d\tau \qquad (2.1)$$

Here D_1 = volume of the solid;

D = volume of cavity;

ρ_1 and $\mathbf{V}_1'$ = density and velocity of points of the body;

ρ and $\mathbf{V}'$ = density and velocity of a particle of fluid;

$\mathbf{g}$ = vector of gravitational acceleration;

$\mathbf{r}'$ = radius vector from the axis of rotation to the point of the body or fluid.

In our case the equation of moments (2.1) yields a projection, which differs from the identical zero, only onto the z' axis.

The integrals which enter into (2.1) [returning to dimensional variables according to Formula (1.5)] may be reduced to the following form

$$\left\{\frac{d}{dt}\int_{D_1}(\mathbf{r}'\times\rho_1\mathbf{V}_1')\,d\tau\right\}_{Z'}=cM_1k^2\sigma'^2e^{\sigma't'} \qquad \left(\sigma'=\frac{\sigma}{T}\right) \qquad (2.2)$$

$$\left\{\frac{d}{dt}\int_{D}(\mathbf{r}'\times\rho\mathbf{V}')\,d\tau\right\}_{Z'}=c\left(Ml'^2\sigma'^2+\sqrt{\nu}\,MQ\sigma'^{\frac{3}{2}}\right)e^{\sigma't'} \qquad \left(MQ=2\pi\rho\int_{\beta_1'}^{\beta_2'}r'^3(\beta')\,d\beta'\right) \qquad (2.3)$$

$$\left\{\int_{D_1}(\mathbf{r}'\times\rho_1\mathbf{g})\,d\tau+\int_{D}(\mathbf{r}'\times\rho\mathbf{g})\,d\tau\right\}_{Z'}=-cg\,(M_1l_1'+Ml')\,e^{\sigma't'} \qquad (2.4)$$

Here M_1 = mass of the solid;

M = mass of the fluid;

k = radius of inertia of the solid relative to the axis of suspension of the pendulum;

l' = distance from the axis of suspension to the center of mass of the cavity;

l_1' = distance from the axis of suspension to the center of gravity of the solid;

β_1', β_2' = coordinates of the poles of the cavity.

Substituting (2.2), (2.3), and (2.4) into (2.1) we obtain the following equation to define σ:

$$\sigma'^2+\lambda\sqrt{\nu}\,\sigma'^{\frac{3}{2}}+\omega^2=0 \qquad \left(\lambda=\frac{MQ}{M_1k^2+Ml'^2},\ \omega^2=g\frac{M_1l_1'+Ml'}{M_1k^2+Ml'^2}\right) \qquad (2.5)$$

We will set $\sigma=\nu\lambda^2x$. Then (2.5) takes the form:

$$x^2 + x^{\frac{3}{2}} + \frac{\omega^2}{\nu^2\lambda^4} = 0 \text{ when } y^4 + y^3 + q^2 = 0 \quad \left(\sqrt{x} = y, \frac{\omega^2}{\nu^2\lambda^4} = q^2\right) \tag{2.6}$$

We will prove that this equation has only two roots which satisfy the condition Re $y > 0$. We set $y = \alpha + i\beta$.

After substitution in (2.6), we obtain for the determination of α and β the system

$$\begin{aligned} (\alpha^2 - \beta^2)^2 - 4\alpha^2\beta^2 + \alpha^3 - 3\alpha\beta^2 + q^2 &= 0 \\ 4\alpha^3 - 4\alpha\beta^2 - \beta^2 + 3\alpha^2 &= 0 \end{aligned} \tag{2.7}$$

Determining β^2 from the second equation of System (2.7) and substituting it in the first equation we obtain:

$$64\alpha^6 + 96\alpha^5 + 48\alpha^4 + 8\alpha^3 - q^2(16\alpha^2 + 8\alpha + 1) = 0, \quad \beta_{1,2} = \pm\alpha\sqrt{\frac{4\alpha+3}{4\alpha+1}} \tag{2.8}$$

The first equation in (2.8) has only one change of sign of the coefficients. According to Descart's rule this polynomial has one positive root. We are not interested in negative roots since in this case the condition Re $\sqrt{x} > 0$ is violated. The second equation of (2.8) gives two values of β which differ only in sign. Therefore, Eq. (2.6) has only two roots for the case Re $y > 0$.

We will solve Eq. (2.6) for the two limiting cases: when q is large and when q is small. Let us recall that the solution of the problem concerning the motion of a fluid within a pendulum cavity was obtained for a large Reynolds number. In turn, the magnitude of the Reynolds number depends essentially on the characteristic period of one oscillation $\pi / |\text{Im } \sigma'|$ which is determined only after solution of Eq. (2.6). Consequently, of all the possible solutions of Eq. (2.6) which correspond to different parameters of the pendulum it is possible to use only those which guarantee a sufficiently large Reynolds number. Later we will see that in the limiting cases of large and small q which we examined, it is always possible to select

pendulum parameters such that the Reynolds number remains large.

Let us examine the case of large q. We will seek the solution of Eq. (2.6) in the form of a series in powers of $1/\sqrt{q}$

$$y = \sqrt{q}\left(y_0 + \frac{1}{\sqrt{q}} y_1 + \dots\right) \tag{2.9}$$

Substituting (2.9) in (2.6) and equating to zero the sum of the coefficients with identical powers of $1/\sqrt{q}$ we obtain equations. the determination of the unknowns y_0, y_1, etc.

For y_0 we have:

$$y_0^4 + 1 = 0, \qquad \operatorname{Re} y_0 > 0$$

whence we obtain:

$$y_{01} = \frac{\sqrt{2}}{2}(1 + i), \qquad y_{02} = \frac{\sqrt{2}}{2}(1 - i)$$

For the determination of y_1 we have

$$4y_1 + 1 = 0 \quad \text{when} \quad y_1 = -{}^1/_4$$

With an accuracy to value of the order of $1/\sqrt{q}$ inclusive, we find

$$y = \frac{\sqrt{2q}}{2} - \frac{1}{4} \pm i\frac{\sqrt{2q}}{2}$$

or

$$\sigma' = -\sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}} \pm i\left(\omega - \sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}}\right) \tag{2.10}$$
$$(\sigma' = \nu\lambda^2 y^2)$$

Thus, in the case under consideration the frequency n and amplitude A of the pendulum oscillations will be, respectively,

$$n = \omega - \sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}}, \qquad A = \exp\left(-\sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}} t'\right)$$

As $\nu \to 0$, the damping factor approaches zero while the oscillation frequency approaches the frequency ω which corresponds to the oscillation of a pendulum with an ideal fluid. We will calculate by what amount the amplitude decreases during each period $T = \pi/|\operatorname{Im}\sigma'|$. Denoting this amount by Δ we have

$$\Delta = 1 - \frac{\exp \operatorname{Re}(\sigma' t_2')}{\exp \operatorname{Re}(\sigma' t_1')} = 1 - \exp \frac{\pi \operatorname{Re} \sigma'}{|\operatorname{Im} \sigma'|} \qquad (2.11)$$

Determining Re σ' and |Im σ'| from (2.10) and substituting in (2.11) we obtain:

$$\Delta = 1 - \exp\left(-\frac{\pi}{2\sqrt{2q-1}}\right) \approx \frac{\pi}{2\sqrt{2q}} \qquad (\Delta \to 0 \text{ when } q \to \infty)$$

It is easy to see that the examined case of large q corresponds to oscillations close to those of a pendulum with an ideal fluid.

Let us consider the simplest example. Let the pendulum be a weightless spherical shell filled with a viscous fluid and connected to the suspension axis by a weightless rod (Fig. 3). For such a pendulum we will have:

$$\lambda = \frac{2R}{l^2}, \quad \omega^2 = \frac{g}{l}, \quad q = \frac{\sqrt{gl^3}}{4\nu}\frac{l^2}{R^2}$$

$$\sigma' \doteq -\sqrt{\nu}\frac{R\sqrt[4]{g}}{l^2\sqrt[4]{4l}} \pm i\left(\sqrt{\frac{g}{l}} - \sqrt{\nu}\frac{R\sqrt[4]{g}}{l^2\sqrt[4]{4l}}\right) \qquad (2.12)$$

The characteristic period of one oscillation with an accuracy to values of the order of $1/\sqrt{q}$ is

$$T \approx \pi\sqrt{l/g} \qquad (2.13)$$

The magnitude of T increases with an increase in the length l of the pendulum. In this case there is the danger of an inadmissible reduction of the Reynolds number. We will set the Reynolds number $N_{Re} \geq 10^4$ and evaluate the allowable pendulum lengths l for various values of radius R. By definition $N_{Re} = R^2/\nu T$.

In order to keep $N_{Re} \geq 10^4$ it is required that

$$T < \frac{R^2}{\nu 10^4} \quad \text{or} \quad \frac{l}{R} < \frac{R^3 g}{\nu^2 \pi^2 10^8}$$

Suppose that the pendulum is filled with water at a temperature of 20°C ($\nu = 1.01 \cdot 10^{-6}$ m²/sec). For R = 0.1 m we have $l/R \leq 10$, i.e., $l \leq 1$ m. For R = 1 m we obtain $l/R \leq 10^4$, i.e., $l \leq 10$ km. With an increase in R the allowable value of l/R increases as R^3. Thus the range of applicability of solution (2.12) is sufficiently

broad.

Let us now examine the case of extremely small q. We note that it is possible to realize small q not only by increasing the kinematic viscosity ν, which might lead to a reduction in N_{Re}. Actually, in the case of the pendulum just considered it is apparent that this may be achieved by decreasing the length of the pendulum l or increasing the radius of the cavity R.

Let us represent Eq. (2.6) in the following form:

$$y^3 (y + 1) = -q^2 \tag{2.14}$$

From this it follows that

$$|y^3|\,|y + 1| = |-q^2| = q^2$$

The product $|y^3|\,|y+1|$ decreases with decreasing q. In this process the factor $|y+1|$ cannot approach zero, otherwise the condition $\mathrm{Re}\, y > 0$ is violated. Consequently the modulus of the unknown root has an order of smallness $q^{2/3}$. Starting from this we will seek the solution of Eq. (2.6) in the form of the following series:

$$y = q^{2/3} (y_0 + q^{2/3} y_1 + \ldots)$$

For the determination of the unknowns y_0 and y_1 we obtain

$$y_0^3 + 1 = 0, \qquad y_0^2 + 3y_1 = 0$$

From this we find:

$$y_{01} = \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}, \quad y_{02} = \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}, \quad y_{11} = \tfrac{1}{6} - i\tfrac{\sqrt{3}}{6}, \quad y_{12} = \tfrac{1}{6} + i\tfrac{\sqrt{3}}{6}$$

With an accuracy to values of the order $q^{2/3}$ inclusive, we have:

$$y = \tfrac{1}{2} q^{2/3} [(1 + \tfrac{1}{3} q^{2/3}) \pm i\sqrt{3}\,(1 - \tfrac{1}{3} q^{2/3})]$$

For σ' we obtain the expression:

$$\sigma' = \frac{\omega^{4/3}}{2\nu^{1/3}\lambda^{2/3}}\left[-\left(1 - \frac{4}{3}\frac{\omega^{2/3}}{\nu^{1/3}\lambda^{2/3}}\right) \pm i\frac{\sqrt{3}}{2}\right] \tag{2.15}$$

or, limiting ourselves to magnitudes of the order $q^{4/3}$:

$$\sigma' = \left(\frac{\omega^4}{\nu\lambda^2}\right)^{1/3}\left(-\frac{1}{2} \pm i\sqrt{3}\right)$$

Just as in the preceding case, we determine by what amount the amplitude drops from its original value during the characteristic period T.

We have

$$\Delta = 1 - \exp \pi \operatorname{Re} \sigma' / |\operatorname{Im} \sigma'|$$

Using Eq. (2.15) to determine Re σ' and |Im σ'|, we obtain

$$\Delta = 1 - \exp\left[-\frac{\pi}{\sqrt{3}}\left(1 - \frac{4}{3} q^{1/3}\right)\right] \qquad (2.16)$$

$$\left(\Delta \to 1 - \exp\left(-\frac{\pi}{\sqrt{3}}\right) \approx 0.84 \text{ when } q \to 0\right)$$

It is apparent that this case differs greatly from the case of oscillations of a pendulum with an ideal fluid. It turns out that it is possible to select a pendulum cavity so large or a pendulum length so small that the influence of viscosity on the motion of the pendulum cannot be neglected despite large Reynolds numbers. In the case of extremely small q the amplitude may drop on the order of 84% from its original value during a single swing.

We will demonstrate, using as our example the pendulum shown in Fig. 3, that by increasing the radius of the cavity R while keeping l and ν constant it is possible to reach as large N_{Re} and as small q as desired. For this pendulum we have:

$$\sigma' = \left(\frac{g^2 l^2}{4R_\nu^2}\right)^{1/3}\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \qquad (2.17)$$

$$q = \frac{\sqrt{g l^3}}{4\nu}\frac{l^2}{R^3}, \qquad N_{Re} = \frac{\sqrt{3}}{2\pi}\frac{R^{4/3}}{\nu}\left(g^2\frac{l^2}{4\nu}\right)^{1/3} \qquad (2.18)$$

It is easily seen from (2.18) that $N_{Re} \to \infty$ and $q \to 0$ as $R \to \infty$. Oscillations of the pendulum in this case are extremely different from oscillations of a similar pendulum with an ideal fluid. Thus, a pendulum filled with water at 20°C ($\nu = 1.01 \cdot 10^{-6}$ m²/sec) with a cavity radius R = 0.1 m and a length l = 0.00016 m will complete 5 oscillations per second, while the same pendulum with an ideal fluid will make 25 oscillations per second. For the pendulum parameters indicated above we will have:

$$q \approx 0.0012, \quad N_{Re} \approx 10^4$$

which completely justifies the use of Solution (2.17) in this case.

Submitted January 29, 1962

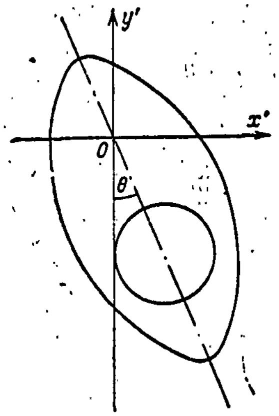

Fig. 1.

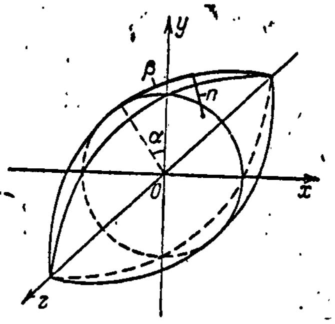

Fig. 2.

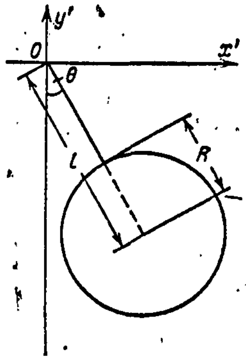

Fig. 3.

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